# Math 601 Final (sample test)

**Name**: ______________________

**This exam has 11 questions, for a total of 150 points.**

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 20 | |
| 8 | 20 | |
| 9 | 15 | |
| 10 | 15 | |
| 11 | 10 | |
| Total: | 150 | |

**Question 1. (10 pts)**

(a) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{bmatrix}$. The eigenvalues of $A$ are 1, 2 and 3. Is $A$ diagonalizable? Justify your answer.

**Solution:** $A$ is a $3 \times 3$ matrix with 3 distinct eigenvalues, so $A$ is diagonalizable.

(b) Suppose $F : \mathbb{R}^4 \to \mathbb{R}^7$ is linear mapping. Can $F$ be surjective? Justify your answer.

**Solution:** We have the following formula

$$\dim R^4 = \dim \operatorname{Ker} F + \dim \operatorname{Im} F$$

Therefore $\dim \operatorname{Im} F \leq 4$. But the dimension $\mathbb{R}^7$ is 7, and $4 < 7$. So $F$ cannot be surjective.

**Question 2. (10 pts)**

Determine whether

$$f(z) = e^{-y}((x+1)\cos x - y\sin(x)) + ie^{-y}((x+1)\sin x + y\cos x)$$

is analytic on $\mathbb{C}$, where $z = x + iy$.

**Solution:** Let

$$u(x, y) = e^{-y}((x+1)\cos x - y\sin(x))$$
$$v(x, y) = e^{-y}((x+1)\sin x + y\cos x)$$

They are clearly differentiable on $\mathbb{C}$. Check Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = e^{-y}(-(1+x)\sin(x) + \cos(x) - y\cos(x))$$
$$\frac{\partial v}{\partial y} = e^{-y}(-(1+x)\sin(x) + \cos(x) - y\cos(x))$$

and

$$\frac{\partial u}{\partial y} = e^{-y}(-(1+x)\cos(x) + y\sin(x) - \sin(x))$$
$$\frac{\partial v}{\partial x} = e^{-y}((1+x)\cos(x) + \sin(x) - y\sin(x))$$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. It follows that $f$ is analytic on $\mathbb{C}$.

**Question 3. (15 pts)**

Evaluate the integral

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz$$

where $C$ is the circle centered at 0 with radius 4.

**Solution:**

$$\frac{e^z}{z^2 - 4z + 3} = \frac{e^z}{(z-1)(z-3)} = \frac{e^z}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

So

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \int_C \frac{e^z}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right) dz$$

Since $C$ is the circle centered at 0 with radius 4, both 1 and 3 are inside $C$. We apply Cauchy's formula and have

$$\int_C \frac{e^z/2}{z-3} dz = 2\pi i(e^3/2)$$

and

$$-\int_C \frac{e^z/2}{z-1} dz = -2\pi i(e/2)$$

So

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \pi i(e^3 - e)$$

**Question 4. (10 pts)**

Evaluate

$$\int_\gamma (3z^2+1)dz$$

where $\gamma$ is the curve $\gamma(t) = (\sin t, t^2+t)$ for $t \in [0,\pi]$, that is, the curve starts at $(0,0)$ and ends at $(0, \pi^2+\pi)$.

**Solution:** Notice that $g(z) = 3z^2+1$ is analytic on $\mathbb{C}$ which is a simply connected domain. Now that two end points of the curve $\gamma$ is 0 and $i(\pi^2+\pi)$. So we have

$$\int_\gamma (3z^2+1)dz = F(i(\pi^2+\pi)) - F(0) = -i(\pi^2+\pi)^3 + i(\pi^2+\pi)$$

where $F(z) = z^3 + z$.

**Question 5. (10 pts)**

Use the residue theorem to evaluate the following integral

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}$$

**Solution:** On the unit circle, we have the identity

$$z = e^{i\theta}$$

Then $\sin\theta = \frac{1}{2i}(z - z^{-1})$. Also, $dz = ie^{i\theta}d\theta$. So

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{(2 - \sin\theta)} &= \int_C \frac{2i}{4i - (z - z^{-1})} \frac{dz}{iz} \\ &= \int_C \frac{2}{4iz - z^2 + 1} dz \\ &= \int_C \frac{-2}{(z - (2+\sqrt{3})i)(z - (2-\sqrt{3})i)} dz \end{aligned}$$

where $C$ is the unit circle. Notice that $(2+\sqrt{3})i$ is outside $C$ and $(2-\sqrt{3})i$ is inside the circle. Write

$$f(z) = \frac{2}{4iz - z^2 + 1}$$

By the residue theorem, we have

$$\int_C \frac{-2}{(z - (2+\sqrt{3})i)(z - (2-\sqrt{3})i)} dz = 2\pi i \ \mathrm{Res}(f, (2-\sqrt{3})i) = 2\pi i \frac{-2}{-2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

So

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \frac{2\pi}{\sqrt{3}}$$

**Question 6. (15 pts)**

Use the residue theorem to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)} dx$$

**Solution:**

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)} dx$$

is the real part of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+4)} dx.$$

So we only need to integrate the latter. We have

$$\frac{e^{iz}}{(z^2+1)(z^2+4)} = \frac{e^{iz}}{(z+i)(z-i)(z+2i)(z-2i)}$$

Note that only $i$ and $2i$ are in the upper half plane. Write

$$f(z) = \frac{e^{iz}}{(z^2+1)(z^2+4)}$$

So the residue theorem implies that

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)(x^2+4)} dx = 2\pi i \left(\text{Res}(f,i) + \text{Res}(f,2i)\right) = \frac{\pi}{6}(2e^{-1} - e^{-2})$$

$$\text{Res}(f,i) = \frac{e^{-1}}{(2i)(3i)(-i)} = \frac{e^{-1}}{6i}$$

$$\text{Res}(f,2i) = \frac{e^{-2}}{(3i)(i)(4i)} = \frac{e^{-2}}{-12i}$$

**Question 7. (20 pts)**

Given the matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Find all eigenvalues of $A$.

**Solution:**

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda)(1-\lambda) + 1 \cdot 2(1-\lambda) \\ &= (\lambda^2 - 5\lambda + 6)(1-\lambda) \end{aligned}$$

So the eigenvalues are $1, 2$ and $3$.

(b) Find a basis for each eigenspace.

**Solution:** When $\lambda = 1$, then

$$A - I = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

whose echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector belonging to the eigenvalue $\lambda = 1$.

The cases for $\lambda = 2$ and 3 are similar.

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector belonging to the eigenvalue $\lambda = 2$.

$$v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector belonging to the eigenvalue $\lambda = 3$.

(c) Determine whether $A$ is diagonalizable. If yes, find an invertible matrix $S$ so that

$$S^{-1}AS$$

is diagonal. If not, explain why.

**Solution:** $A$ is diagonalizable, since $A$ is a $3 \times 3$ matrix with 3 linearly independent eigenvectors. Let

$$S = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then we have $S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**Question 8. (20 pts)**

Let $V$ be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Use Gram-Schmidt process to find an orthonormal basis of $V$.

**Solution:** Note that we are trying to find an **orthonormal** basis. Let

$$w_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $w_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$$v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

So $w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Now $w_1, w_2$ and $w_3$ form an orthonormal basis of $V$.

(b) Find the projection of

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

onto $V$.

**Solution:** Since $w_1, w_2$ and $w_3$ form an **orthonormal** basis of $V$.

$$\mathrm{Proj}_V(x) = \langle x, w_1 \rangle w_1 + \langle x, w_2 \rangle w_2 + \langle x, w_3 \rangle w_3 = \begin{bmatrix} 3/2 \\ -1 \\ 5 \\ 3/2 \end{bmatrix}$$

**Question 9. (15 pts)**

Let $V$ be the vector space spanned by $\{e^x, e^{-x}, xe^x, xe^{-x}\}$. Accept as a fact that

$$e^x,\ e^{-x},\ xe^x,\ xe^{-x}$$

form a basis for $V$. Let us denote this basis by $\mathfrak{B}$. Let

$$T(f) = 2f - f'$$

be a linear transformation from $V$ to $V$.

(a) Find the $\mathfrak{B}$-matrix of $T$.

**Solution:**

$$T(e^x) = 2e^x - e^x = e^x$$

similarly, we have

$$T(e^{-x}) = 3e^{-x}$$
$$T(xe^x) = xe^x - e^x$$
$$T(xe^{-x}) = 3xe^{-x} - e^{-x}$$

So

$$[T]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) Is $T$ an isomorphism?

**Solution:** $T$ is an isomorphism, since the determinant of $[T]_{\mathfrak{B}} = 1 \cdot 3 \cdot 1 \cdot 3 = 9 \neq 0$.

**Question 10. (15 pts)**

Given

$$A = \begin{bmatrix} 2 & 2 & -3 & 1 & 13 \\ 1 & 1 & 1 & 1 & -1 \\ 3 & 3 & -5 & 0 & 14 \\ 6 & 6 & -2 & 4 & 16 \end{bmatrix}$$

(a) Find a basis of $\mathrm{Ker}(A)$.

**Solution:** First, use elementary row operations to get the reduced row echelon form of $A$.

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So all elements in $\mathrm{Ker}\, A$ are of the form

$$t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

So

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

form a basis of the kernel.

(b) Find a basis of the row space of $A$.

**Solution:** The three nonzero rows in the reduced row echelon form of $A$ form a basis of the row space of $A$. That is

$$u_1 = (1, 1, 0, 0, -2)$$

$$u_2 = (0, 0, 1, 0, -4)$$

$$u_3 = (0, 0, 0, 1, 5)$$

form a basis of the row space of $A$.

**Question 11. (10 pts)**

Suppose $\lambda$ is an eigenvalue of an $n \times n$-matrix $A$.

(a) Show that $\lambda^n$ is an eigenvalue of $A^n$.

**Solution:** Since $\lambda$ is an eigenvalue of $A$, then there exists a nonzero vector $v$ such that

$$Av = \lambda v.$$

It follows that

$$A^2(v) = A(Av) = A(\lambda v) = \lambda Av = \lambda^2 v$$

By induction, we see that

$$A^n(v) = \lambda^n v$$

So $\lambda^n$ is an eigenvalue of $A^n$.

(b) Consider the matrix

$$B = c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I_n$$

where $c_i$ are real numbers. Show that the real number

$$\mu = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0$$

an eigenvalue of $B$.

**Solution:** Let $v$ be the same vector from part (a). Then by part (a), we have

$$\begin{aligned} B(v) &= (c_n A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I_n)(v) \\ &= c_n A^n(v) + c_{n-1} A^{n-1}(v) + \cdots + c_1 A(v) + c_0 v \\ &= (c_n \lambda^n + \cdots c_1 \lambda + c_0) v = \mu v \end{aligned}$$

So $\mu$ is an eigenvalue of $B$.